How Sensitive are Tail-Related Risk Measures in a Contamination Neighbourhood?

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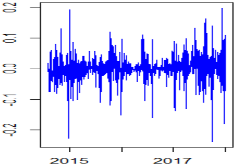
Motivation

- Financial asset returns often contain observations that are inconsistent with the majority of the data
- Estimation or mis-specification errors in the portfolio loss distribution can have a considerable impact on risk measures
- Choosing risk measures and its risk level plays an important role in financial practice and risk management

Example

CRIX, the benchmark cryptocurrency index by Trimborn and Härdle (2016)



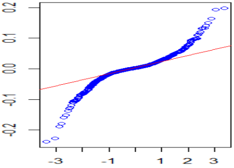


Figure 1: Time series (left) and normal Q-Q plot (right) of the daily log returns of CRIX during 20140731-20180101, standardized by using GARCH(1, 1) model. Data source: crix.berlin

Common risk measures

- **Value-at-Risk** (VaR), the minimal loss of the α worst cases
- **Expected-shortfall** (ES), the expectation of the α worst loss
- **Expectile**, one-to-one mapping with VaR, reflects the tail heaviness through the expectile-quantile transformation
  - Expectile level w_α such that $e_{w_\alpha} = q_\alpha$, satisfies ▸ Details

$$w_\alpha = \frac{\int_{-\infty}^{u} x dF(x) - u\alpha}{2\{\int_{-\infty}^{u} x dF(x) - u\alpha\} + u - \mathrm{E}[X]}, \quad u = q_\alpha \qquad (1)$$

  - ES using expectile, Taylor (2008)

$$ES_\alpha = e_{w_\alpha} + \frac{e_{w_\alpha} - \mathrm{E}[X]}{1 - 2w_\alpha} \frac{w_\alpha}{\alpha}$$

Example

Consider a mixture model given in Kuan et al. (2009)

$$F_\epsilon(x) = (1-\epsilon)\Phi(x\sqrt{1-\epsilon}) + \epsilon\Phi(x\sqrt{\epsilon})$$

- Contamination level ϵ
- Heaviness parameter $\sigma = 1/\sqrt{\epsilon}$
- Interplay of risk level α and contamination level ϵ to determine risk measures

Research Questions

- How to establish suitable contamination models for such data?
- How sensitive are risk measures for such models?
- How are risk measures adjusted for contamination data?

Outline

1. Motivation ✓
2. Contamination models
3. Approximation results
4. Numerical examples
5. Empirical study
6. Conclusions

Contamination Models

Consider

$$F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x), \quad x \in \mathbb{R},\ \epsilon \in [0,1]$$

- F stands for the pre-supposed ideal model
- H represents plausible deviations from F
- ϵ reflects the amount of uncertainty in F

Common Assumptions

- Both F and H have infinite left endpoints since we are interested with infinite risks
- Catastrophe contamination attracts financial regulators and thus assume H has a heavier tail than F
- Risk measures concern extreme value risk management and thus suppose F and H are common distributions in EVT

Notation

- Recall that F and F_ϵ stand for the pre-supposed ideal model and the contamination model in the ϵ neighbourhood of F
- Denote by ϱ_α and $\varrho_\alpha(\epsilon)$ the risk measure ϱ of F and F_ϵ
  - Value-at-Risk (VaR): q_α and $q_\alpha(\epsilon)$
  - Expected shortfall (ES): ES_α and $ES_\alpha(\epsilon)$
  - Expectile-quantile transformation level: w_α and $w_\alpha(\epsilon)$

Approximation of Risk Measures

- **Case A**: common contamination with $\epsilon \in (0, 1]$ fixed and $\alpha \to 0$
- **Case B**: interplay between ϵ and α with $\epsilon = \epsilon_\alpha \to 0$ as $\alpha \to 0$
- **Case C**: infinitestimal contamination with $\alpha \in (0, 1)$ fixed and $\epsilon \to 0$

Case A: Fixed ϵ

Theorem A *Let* $F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon H(x)$ *with* $\epsilon \in (0,1]$. *We have as* $\alpha \to 0$

$$q_\alpha(\epsilon) \sim q_{\alpha'}(1) \quad \textit{with} \quad \alpha' \stackrel{\text{def}}{=} \alpha/\epsilon$$

Further, if both exist with finite means, then

$$ES_\alpha(\epsilon) \sim ES_{\alpha'}(1), \quad \frac{w_\alpha(\epsilon)}{\alpha} \sim \frac{w_{\alpha'}(\epsilon)}{\alpha'}$$

such that $e_{w_\alpha(\epsilon)}(\epsilon) = q_\alpha(\epsilon)$

Case B: Dynamic ϵ

Theorem B *Suppose* $\epsilon = \epsilon_\alpha \to 0$ *as* $\alpha \to 0$, *we have*
a) If $F(x)/\{\epsilon H(x)\} \to 0$ *as* $\alpha \to 0$ *with* x *the* α *quantile of* F_ϵ, *then*

$$q_\alpha(\epsilon) \sim q_{\alpha/\epsilon}(1), \quad ES_\alpha(\epsilon) \sim ES_{\alpha'}(1), \quad \frac{w_\alpha(\epsilon)}{\alpha} \sim \frac{w_{\alpha'}(\epsilon)}{\alpha'}$$

b) If $\epsilon H(x)/F(x) \to 0$ *as* $\alpha \to 0$ *with* x *the* α *quantile of* F_ϵ, *then*

$$q_\alpha(\epsilon) \sim q_\alpha, \quad ES_\alpha(\epsilon) \sim ES_\alpha, \quad w_\alpha(\epsilon) \sim w_\alpha$$

Case C: Independent Small ϵ

Define the influence function (IF) of risk measure ϱ of F_ϵ as

$$IF(\varrho; F, H) = \lim_{\epsilon \to 0} \frac{\varrho(\epsilon) - \varrho}{\epsilon} = \frac{\partial \varrho(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$$

Hence, approximations of $\varrho(\epsilon)$ with infinitestimal contamination level ϵ is given by

$$q_\alpha(\epsilon) \simeq q_\alpha + \epsilon IF(q_\alpha; F, H), \qquad ES_\alpha(\epsilon) \simeq ES_\alpha + \epsilon IF(ES_\alpha; F, H)$$

Case C: Influence Function

Theorem C *Assume that F has positive continuous differential at its α quantile, and H is continuous at q_α. We have*

$$IF(q_\alpha; F, H) = \frac{\alpha - H(q_\alpha)}{F'(q_\alpha)}$$

$$IF(ES_\alpha; F, H) = \frac{q_\alpha\{\alpha - H(q_\alpha)\} + \int_{-\infty}^{q_\alpha} x \, \mathrm{d}\{H(x) - F(x)\}}{\alpha}$$

Example

Example 1 Normal contamination

Consider $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon\Phi(x/\sigma)$ with $\sigma > 1$, the scale parameter of the contamination model

- The larger the σ is, the heavier the contamination model is
- For $\epsilon \in (0,1)$, we have $\varrho_\alpha(\epsilon) \sim \varrho_{\alpha/\epsilon}$
- Let $\epsilon = \alpha^\tau$ with $\tau > 0$ a constant. The larger τ is, the lower the contamination level is, and thus the risk measure might be more robustness

Example

Example 2 Laplace contamination
Consider $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon L(x/\sigma)$ with $\sigma > 0$ and L the standard double-sided exponential distribution

$$L(x) = \frac{1}{2}\exp\left\{-\sqrt{2}|x|\right\}, \quad x \in \mathbb{R}$$

- Essential heavier of Laplace model than normal model
- Approximations based on L might be faster than normal contamination
- In order to obtain non-sensitivity of risk measures, a very smaller $\epsilon = \alpha^\tau$ (and thus a larger τ) is needed

Example

Example 3 Power-like contamination

Consider $F_\epsilon(x) = (1-\epsilon)\Phi(x) + \epsilon H(x/\sigma)$ with $\sigma > 0$ and H a symmetry distribution with

$$H(x) = \frac{1}{2}\left\{1 - \left(1 - \frac{4}{4+x^2}\right)^{0.5}\right\}, \quad x < 0$$

- Power decaying tail and infinite variance, not useful in practice
- Typical example with expectile coinciding with quantile
- All of the following results are obtained by R codes at

SRMC

Laplace: $\epsilon = 0.5, \sigma = 1.6$

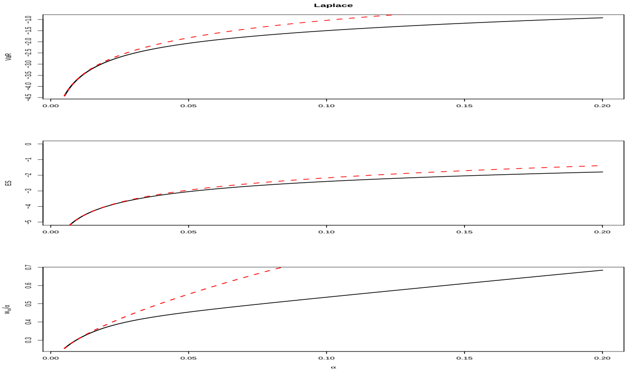


Figure 2: Approximations based on $N(0,1)$ to its **true values**, indicated by dotted line and black line

Laplace: $\alpha = 0.5\%, \sigma = 1$

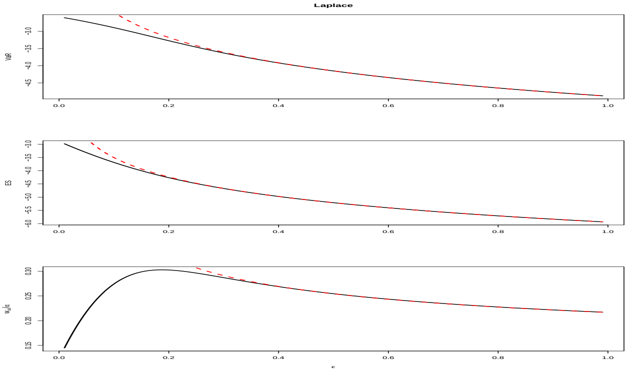


Figure 3: Approximations based on $N(0, 1)$ to its **true values**, indicated by dotted line and black line

Laplace: $\tau = 1, \sigma = 1.6$

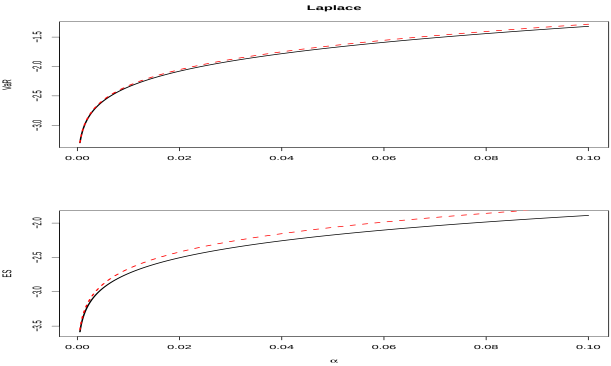


Figure 4: Approximations based on $N(0,1)$ to its **true values**, indicated by dotted line and black line

Laplace: $\tau = 0.1, \sigma = 0.95$

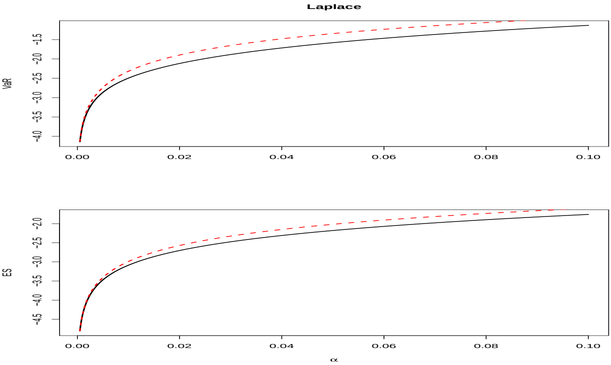


Figure 5: Approximations based on $L(\sigma = 0.95)$ to its **true values**, indicated by dotted line and black line

RE Based on IF

Approximate of VaR and ES for F_ϵ with small ϵ as

$$\widetilde{\varrho}(\epsilon) \stackrel{\text{def}}{=} \varrho + \epsilon IF(\varrho; F, H)$$

Define thus the relative error (RE) as

$$\text{RE}(\varrho) = \frac{\widetilde{\varrho}(\epsilon) - \varrho(\epsilon)}{\varrho(\epsilon)}$$

◂ Influence function

RE Based on IF

Normal: $\sigma = 2$					
ϵ	0.10	2.10	4.10	6.10	8.10
RE(q_α)	−0.02	−0.01	0.01	0.04	0.08
RE(ES_α)	0.01	0.01	0.01	0.02	0.03
Laplace: $\sigma = 1.2$					
ϵ	0.10	2.10	4.10	6.10	8.10
RE(q_α)	−0.02	−0.01	0.02	0.06	0.11
RE(ES_α)	0.01	0.01	0.01	0.02	0.03
Power-like: $\sigma = 1$					
ϵ	0.10	2.10	4.10	6.10	8.10
RE(q_α)	−0.02	−0.11	−0.33	−0.71	−1.22
RE(ES_α)	0.01	0.03	0.08	0.15	0.26

Table 1: RE of VaR and ES with $\alpha = 0.25$ is in ‰

Tail Analyses via MEF

- Empirical Mean Excess (EME) of $X \sim F$ is given as

$$\widehat{m}_X(t) = \frac{\sum_{i=1}^{n} (X_i - t)\,\mathbf{I}\{X_i > t\}}{\sum_{i=1}^{n} \mathbf{I}\{X_i > t\}}, \quad t \text{ large}$$

- **Power-like tails** as $\overline{F}(x) \sim Cx^{-1/\gamma}$ are implied by

$$\widehat{m}_X(t) \sim \frac{\gamma}{1-\gamma} t, \quad \gamma \geq 0$$

- **Weibull-like tails** as $\overline{F}(x) \sim C \exp\{-x^{\tau}\}$ are implied by

$$\log \widehat{m}_X(t) \sim (1-\tau) \log t, \quad \tau > 0$$

CRIX: Tail Heaviness Exploration

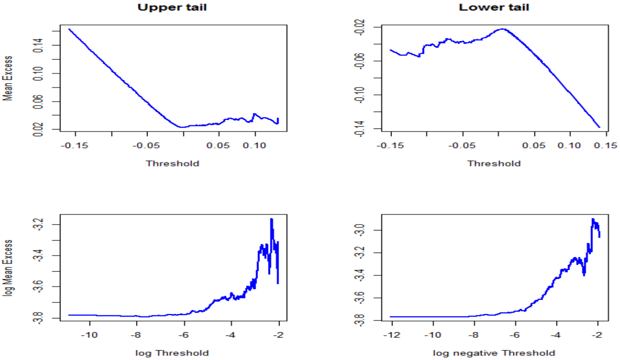


Figure 6: EME and log EME for daily log returns of CRIX during 20140731–20180101. Lower tail of X is given by the upper tail of $-X$

Model Choice of CRIX

- Data appears with Laplace tails
- The principal data can be modeled by normal distribution
- Consider the normal-Laplace contamination model with parameter $\epsilon, \boldsymbol{\mu} = (\mu_1, \mu_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$

$$F_\epsilon(x) = (1-\epsilon) * \Phi\left(\frac{x-\mu_1}{\sigma_1}\right) + \epsilon * L\left(\frac{x-\mu_2}{\sigma_2}\right), \quad x \in \mathbb{R}$$

Estimation of CRIX

- EM algorithm to estimate parameters involved
- Three periods are considered
  - A, 20140731–20180101
  - B, 20140731–20160401
  - C, 20160402–20180101
- Three methods to estimate VaR and ES
  - Historical simulations: $\widehat{q}^*_\alpha, \widehat{ES}^*_\alpha$
  - Laplace approximation: $\widehat{q}_{\alpha'}(1), \widehat{ES}_{\alpha'}(1)$
  - Complete mixture model: $\widehat{q}_\alpha(\epsilon), \widehat{ES}_\alpha(\epsilon)$

Estimation of CRIX

per.	ϵ	μ_1	μ_2	σ_1	σ_2
A	0.622	0.002	0.004	0.010	0.045
B	0.480	0.001	−0.002	0.014	0.045
C	0.731	0.002	0.008	0.006	0.045

Table 2: Estimated parameters of normal-Laplace contamination

Estimation of CRIX

per.	$\alpha(\%)$	$\widehat{q}^*_\alpha$	$\widehat{q}_{\alpha'}(1)$	$\widehat{q}_\alpha(\epsilon)$	$\widehat{ES}^*_\alpha$	$\widehat{ES}_{\alpha'}(1)$	$\widehat{ES}_\alpha(\epsilon)$
A	0.5	0.136	0.127	0.127	0.183	0.125	0.000
	1	0.105	0.105	0.105	0.152	0.138	0.000
	5	0.054	0.055	0.054	0.091	0.085	0.000
B	0.5	0.118	0.125	0.125	0.171	0.184	0.000
	1	0.104	0.103	0.103	0.143	0.130	0.000
	5	0.046	0.052	0.052	0.086	0.094	0.000
C	0.5	0.137	0.128	0.128	0.179	0.128	0.000
	1	0.108	0.106	0.106	0.155	0.130	0.000
	5	0.059	0.055	0.055	0.095	0.080	0.000

Table 3: Estimated VaR, ES based on normal-Laplace model

Conclusions

- Common data-sets such as CRIX follow certain contamination models in Huber's framework
- Practitioners must model carefully the tail feature in risk management
- Theoretical approximations of risk measures are given with illustrated examples

Further work

- How to model the *dynamic* tail features of the data-sets?
- What about the *min-max* problem for these risk measures?
- How about the *statistical estimations* of these models?

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Expectile and Quantile

- Both are elicitable, since e_α and q_α correspond to ϱ_α^2 and ϱ_α^1, respectively

$$\varrho_\alpha^\gamma = \arg\min_x \alpha \, \mathsf{E}\left[(X-x)_+^\gamma\right] + (1-\alpha)\,\mathsf{E}\left[(X-x)_-^\gamma\right]$$

- First-order condition: setting $x = e_\alpha, q_\alpha$ subsequently

$$\alpha \, \mathsf{E}\left[(X-x)_+\right] = (1-\alpha)\,\mathsf{E}\left[(X-x)_-\right] \tag{2}$$

$$\alpha \mathbb{P}\{X \geq x\} = (1-\alpha)\mathbb{P}\{X \leq x\}$$

Expectile and Quantile

Specifying $x = e_{w_\alpha}$ in (2), we have

$$w_\alpha = \frac{\mathrm{E}\left[(X-x)_-\right]}{\mathrm{E}\left[(X-x)_-\right] + \mathrm{E}\left[(X-x)_+\right]}$$
$$= \frac{\mathrm{E}\left[(X-x)_-\right]}{2\,\mathrm{E}\left[(X-x)_-\right] + \mathrm{E}\left[X\right] - x}, \quad \text{by using } y_+ = y + y_-$$

with

$$\mathrm{E}\left[(X-x)_-\right] = xF(x) - \int_{-\infty}^{x} y dF(y)$$

The desired expression w_α in (1) is thus obtained since $F(x) = \alpha$ due to $e_{w_\alpha} = q_\alpha$ Common risk measure